GOVERNMENT DEGREE COLLEGE

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LINEAR ALGEBRA AND IT'S APPLICATIONS

A PROJECT REPORT SUBMITTED

BY

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UNDER THE ESTEEMED GUIDANCE OF

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IN

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DEPARTMENT OF MATHEMATICS

GOVERNMENT DEGREE COLLEGE

RAMPACHODAVARAM
ANDRAPRADESH,INDIA

BONAFIDE CERTIFICATE

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LINEAR ALGEBRA AND IT'S APPLICATIONS

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TO

DEPARTMENT OF MATHEMATICS

VECTOR CALCULUS AND ITS APPLICATIONS

ABSTRACT

Abstract this chapter sets the ground for the derivation of the conservation equations by providing a brief review of the continuum mechanics tools needed for that purpose while establishing some of the mathematical notations and procedures that will be used throughout the book. The review is by no mean comprehensive and assumes a basic knowledge of the fundamentals of continuum mechanics. A short introduction of the elements of linear algebra including vectors, matrices, tensors, and their practices is given. The chapter ends with an examination of the fundamental theorems of vector calculus, which constitute the elementary building blocks needed for manipulating and solving these conservation equations either analytically or numerically using computational fluid dynamics.

CONTENTS

TOPIC ***PAGENUMBERS***

Introduction :

The transfer phenomena of interest here can be mathematically represented by equations involving physical variables that fall under three categories: scalars, vectors, and tensors [1–3]. Throughout this book scalars are designated by lightface italic, vectors by lower boldface Roman, and tensors by boldface Greek letters. In addition, matrices are identified by upper boldface Roman letters. A scalar represents a quantity that has magnitude such as volume V, pressure p, temperature T, time t, mass m, and density q. A vector represents a quantity of a given magnitude and direction such as velocity v, momentum L ¼ mv, and force F. A matrix is a rectangular array of quantities ordered along rows and columns. A tensor is a mathematical object analogous to but more general than a vector, represented by an array of components, such as the shear stress tensor. Moreover, the conservation equations are composed of terms that represent the product of two or more variables. The multiplication involved may be of various types to be detailed later and the variables could be a combination of the three types described above. Whenever the multiplication results in a scalar, the product will be enclosed by parentheses "(product)", if it results in a vector it will be enclosed by square brackets "[product]", and if it results in a tensor it will be enclosed by curly brackets "{product}".

Vector calculus, or vector analysis, is a branch of mathematics concerned with differentiation and integration of vector fields, primarily in 3-dimensional Euclidean space The term "vector calculus" is sometimes used as a synonym for the broader subject of multivariable calculus, which includes vector calculus as well as partial differentiation and multiple integration. Vector calculus plays an important role in differential geometry and in the study of partial differential equations. It is used extensively in many disciplines, such as physics, engineering, and machine learning. Example applications include electromagnetic fields, gravitational fields, fluid flow, and back propagation.

Vector calculus was developed from quaternion analysis by J. Willard Gibbs and Oliver Heaviside near the end of the 19th century, and most of the notation and terminology was established by Gibbs and Edwin Bidwell Wilson in their 1901 book, Vector Analysis. In the conventional form using cross products, vector calculus does not generalize to higher dimensions, while the alternative approach of geometric algebra, which uses exterior products does generalize, as discussed below.

DEFINATIONS :

Scalar fields :

A scalar field associates a scalar value to every point in a space. The scalar may either be a mathematical number or a physical quantity. Examples of scalar fields in applications include the temperature distribution throughout space, the pressure distribution in a fluid, and spin-zero quantum fields, such as the Higgs field. These fields are the subject of scalar field theory.

Vector fields :

A vector field is an assignment of a vector to each point in a subset of space.[1] *A vector field in the plane, for instance, can be visualized as a collection of arrows with a given magnitude and direction each attached to a point in the plane. Vector fields are often used to model, for example, the speed and direction of a moving fluid throughout space, or the strength and direction of some force, such as the magnetic or gravitational force, as it changes from point to point.*

Vectors and pseudo vectors :

In more advanced treatments, one further distinguishes pseudo vector fields and pseudo scalar fields, which are identical to vector fields and scalar fields except that they change sign under an orientation-reversing map: for example, the curl of a vector field is a pseudo vector field, and if one reflects a vector field, the curl points in the opposite direction. This distinction is clarified and elaborated in geometric algebra, as described below.

Vector algebra :

The algebraic (non-differential) operations in vector calculus are referred to as vector algebra, being defined for a vector space and then globally applied to a vector field. The basic algebraic operations consist of:

Operation	Notation	Description
Vector addition	$\mathbf{v}_1 + \mathbf{v}_2$	Addition of two vector fields, yielding a vector field.
Scalar multiplication	$a\mathbf{v}$	Multiplication of a scalar field and a vector field, yielding a vector field.
Dot product	$\mathbf{v}_1 \cdot \mathbf{v}_2$	Multiplication of two vector fields, yielding a scalar field.
Cross product	$\mathbf{v}_1 \times \mathbf{v}_2$	Multiplication of two vector fields in $\mathbf{R}^3$, yielding a (pseudo)vector field.

Also commonly used are the two triple products:

Operation	Notation	Description
Scalar triple product	$\mathbf{v}_1 \cdot (\mathbf{v}_2 \times \mathbf{v}_3)$	The dot product of a vector and a cross product of two vectors.
Vector triple product	$\mathbf{v}_1 \times (\mathbf{v}_2 \times \mathbf{v}_3)$	The cross product of a vector and a cross product of two vectors.

Operators and theorems

Differential operators

Main articles: Gradient, Divergence, Curl (mathematics), and Laplacian

Vector calculus studies various differential operators defined on scalar or vector fields, which are typically expressed in terms of the del operator ( ), also known as "nabla". The three basic vector operators are:

Operation	Notation	Description	Notational analogy	Domain/Range
Gradient	$\operatorname{grad}(f) = \nabla f$	Measures the rate and direction of change in a scalar field.	Scalar multiplication	Maps scalar fields to vector fields.
Divergence	$\operatorname{div}(\mathbf{F}) = \nabla \cdot \mathbf{F}$	Measures the scalar of a source or sink at a given point in a vector field.	Dot product	Maps vector fields to scalar fields.
Curl	$\operatorname{curl}(\mathbf{F}) = \nabla \times \mathbf{F}$	Measures the tendency to rotate about a point in a vector field in $\mathbb{R}^3$.	Cross product	Maps vector fields to (pseudo)vector fields.
f denotes a scalar field and F denotes a vector field				

Also commonly used are the two Laplace operators:

Operation	Notation	Description	Domain/Range
Laplacian	$\Delta f = \nabla^2 f = \nabla \cdot \nabla f$	Measures the difference between the value of the scalar field with its average on infinitesimal balls.	Maps between scalar fields.
Vector Laplacian	$\nabla^2 \mathbf{F} = \nabla(\nabla \cdot \mathbf{F}) - \nabla \times (\nabla \times \mathbf{F})$	Measures the difference between the value of the vector field with its average on infinitesimal balls.	Maps between vector fields.
f denotes a scalar field and F denotes a vector field			

A quantity called the Jacobian matrix is useful for studying functions when both the domain and range of the function are multivariable, such as a change of variables during integration

Integral theorems :

The three basic vector operators have corresponding theorems which generalize the fundamental theorem of calculus to higher dimensions:

Theorem	Statement	Description
Gradient theorem	$\int_{L\subset\mathbb{R}^n} \nabla\varphi\cdot d\mathbf{r} = \varphi(\mathbf{q})-\varphi(\mathbf{p}) \quad \text{for} \quad L = L[p\to q]$	The line integral of the gradient of a scalar field over a curve L is equal to the change in the scalar field between the endpoints p and q of the curve.
Divergence theorem	$\underbrace{\int\cdots\int_{V\subset\mathbb{R}^n}}_{n}(\nabla\cdot\mathbf{F})\,dV = \underbrace{\oint\cdots\oint_{\partial V}}_{n-1}\mathbf{F}\cdot d\mathbf{S}$	The integral of the divergence of a vector field over an n-dimensional solid V is equal to the flux of the vector field through the $(n-1)$-dimensional closed boundary surface of the solid.
Curl (Kelvin–Stokes) theorem	$\iint_{\Sigma\subset\mathbb{R}^3}(\nabla\times\mathbf{F})\cdot d\mathbf{\Sigma} = \oint_{\partial\Sigma}\mathbf{F}\cdot d\mathbf{r}$	The integral of the curl of a vector field over a surface Σ in $\mathbb{R}^3$ is equal to the circulation of the vector field around the closed curve bounding the surface.
φ denotes a scalar field and F denotes a vector field		

In two dimensions, the divergence and curl theorems reduce to the Green's theorem:

Theorem	Statement	Description
Green's theorem	$\iint_{A\subset\mathbb{R}^2}\left(\frac{\partial M}{\partial x}-\frac{\partial L}{\partial y}\right)dA = \oint_{\partial A}(L\,dx + M\,dy)$	The integral of the divergence (or curl) of a vector field over some region A in $\mathbb{R}^2$ equals the flux (or circulation) of the vector field over the closed curve bounding the region.
For divergence, $F=(M,-L)$. For curl, $F=(L,M,0)$. L and M are functions of (x, y).		

Gradient theorem :

The gradient theorem, also known as the fundamental theorem of calculus for line integrals, says that a line integral through a gradient field can be evaluated by evaluating the original scalar field at the endpoints of the curve.

Let $\varphi : U \subseteq \mathbb{R}^n \to \mathbb{R}$ and γ is any curve from p to q. Then

$$\varphi(\mathbf{q}) - \varphi(\mathbf{p}) = \int_{\gamma[\mathbf{p},\,\mathbf{q}]} \nabla\varphi(\mathbf{r}) \cdot d\mathbf{r}.$$

It is a generalization of the fundamental theorem of calculus to any curve in a plane or space (generally n-dimensional) rather than just the real line.

The gradient theorem implies that line integrals through gradient fields are path independent. In physics this theorem is one of the ways of defining a conservative force. By placing φ as potential, ∇φ is a conservative field.Work done by conservative forces does not depend on the path followed by the object, but only the end points, as the above equation shows.

The gradient theorem also has an interesting converse: any path-independent vector field can be expressed as the gradient of a scalar field. Just like the gradient theorem itself, this converse has many striking consequences and applications in both pure and applied mathematics.

Pr

If φ is a differentiable function from some open subset U (of $\mathbb{R}^n$) to ℝ, and if r is a differentiable function from some closed interval [a, b] to U, then by the multivariate chain rule, the composite function φ ∘ r is differentiable on (a, b) and

$$\frac{d}{dt}(\varphi \circ \mathbf{r})(t) = \nabla\varphi(\mathbf{r}(t)) \cdot \mathbf{r}'(t)$$

for all t in (a, b). Here the · denotes the usual inner product.

Now suppose the domain U of φ contains the differentiable curve γ with endpoints p and q, (oriented in the direction from p to q).

If r parametrizes γ for t in [a, b], then the above shows that [1]

$$\int_\gamma \nabla\varphi(\mathbf{u})\cdot \mathrm{d}\mathbf{u} = \int_a^b \nabla\varphi(\mathbf{r}(t))\cdot \mathbf{r}'(t)\mathrm{d}t$$
$$= \int_a^b \frac{d}{dt}\varphi(\mathbf{r}(t))\mathrm{d}t = \varphi(\mathbf{r}(b)) - \varphi(\mathbf{r}(a)) = \varphi(\mathbf{q}) - \varphi(\mathbf{p}),$$

where the definition of the line integral is used in the first equality, and the fundamental theorem of calculus is used in the third equality.

Examples

Example 1

Suppose $\gamma \subset \mathbb{R}^2$ *is the circular arc oriented counterclockwise from (5, 0) to (−4, 3). Using the definition of a line integral,*

$$\begin{aligned}
\int_\gamma y\mathrm{d}x + x\mathrm{d}y &= \int_0^{\pi-\tan^{-1}\left(\frac{3}{4}\right)} ((5\sin t)(-5\sin t) + (5\cos t)(5\cos t))\mathrm{d}t \\
&= \int_0^{\pi-\tan^{-1}\left(\frac{3}{4}\right)} 25\left(-\sin^2 t + \cos^2 t\right)\mathrm{d}t \\
&= \int_0^{\pi-\tan^{-1}\left(\frac{3}{4}\right)} 25\cos(2t)\mathrm{d}t \\
&= \frac{25}{2}\sin(2t)\Big|_0^{\pi-\tan^{-1}\left(\frac{3}{4}\right)} \\
&= \frac{25}{2}\sin\left(2\pi - 2\tan^{-1}\left(\frac{3}{4}\right)\right) \\
&= -\frac{25}{2}\sin\left(2\tan^{-1}\left(\frac{3}{4}\right)\right) \\
&= -\frac{25\left(\frac{3}{4}\right)}{\left(\frac{3}{4}\right)^2 + 1} = -12.
\end{aligned}$$

Notice all of the painstaking computations involved in directly calculating the integral. Instead, since the function f(x, y) = xy is differentiable on all of $\mathbb{R}^2$, we can simply use the gradient theorem to say

$$\int_\gamma y\mathrm{d}x + x\mathrm{d}y = \int_\gamma \nabla(xy)\cdot(\mathrm{d}x,\mathrm{d}y) = xy\Big|_{(5,0)}^{(-4,3)} = -4\cdot 3 - 5\cdot 0 = -12.$$

Notice that either way gives the same answer, but using the latter method, most of the work is already done in the proof of the gradient theorem.

Example 2

For a more abstract example, suppose $\gamma \subset \mathbb{R}^n$ has endpoints p, q, with orientation from p to q. For u in $\mathbb{R}^n$, let |u| denote the Euclidean norm of u. If $\alpha \geq 1$ is a real number, then

$$\int_\gamma |\mathbf{x}|^{\alpha-1}\mathbf{x}\cdot d\mathbf{x} = \frac{1}{\alpha+1}\int_\gamma (\alpha+1)|\mathbf{x}|^{(\alpha+1)-2}\mathbf{x}\cdot d\mathbf{x}$$
$$= \frac{1}{\alpha+1}\int_\gamma \nabla|\mathbf{x}|^{\alpha+1}\cdot d\mathbf{x} = \frac{|\mathbf{q}|^{\alpha+1}-|\mathbf{p}|^{\alpha+1}}{\alpha+1}$$

Here the final equality follows by the gradient theorem, since the function $f(x) = |x|^{\alpha+1}$ is differentiable on $\mathbb{R}^n$ if $\alpha \geq 1$.

If $\alpha < 1$ then this equality will still hold in most cases, but caution must be taken if γ passes through or encloses the origin, because the integrand vector field $|x|^{\alpha-1}x$ will fail to be defined there. However, the case $\alpha = -1$ is somewhat different; in this case, the integrand becomes $|x|^{-2}x = \nabla(\log |x|)$, so that the final equality becomes $\log |q| - \log |p|$.

Note that if $n = 1$, then this example is simply a slight variant of the familiar power rule from single-variable calculus.

Example 3

Suppose there are n point charges arranged in three-dimensional space, and the i-th point charge has charge Q_i and is located at position p_i in $\mathbb{R}^3$. We would like to calculate the work done on a particle of charge q as it travels from a point a to a point b in $\mathbb{R}^3$. Using Coulomb's law, we can easily determine that the force on the particle at position r will be

$$\mathbf{F}(\mathbf{r}) = kq\sum_{i=1}^{n}\frac{Q_i(\mathbf{r}-\mathbf{p}_i)}{|\mathbf{r}-\mathbf{p}_i|^3}$$

Here $|u|$ denotes the Euclidean norm of the vector u in $\mathbb{R}^3$, and $k = 1/(4\pi\varepsilon_0)$, where ε_0 is the vacuum permittivity.

Let $\gamma \subset \mathbb{R}^3 - \{p_1, ..., p_n\}$ be an arbitrary differentiable curve from a to b. Then the work done on the particle is

$$W = \int_\gamma \mathbf{F}(\mathbf{r}) \cdot d\mathbf{r} = \int_\gamma \left(kq \sum_{i=1}^{n} \frac{Q_i(\mathbf{r} - \mathbf{p}_i)}{\left|\mathbf{r} - \mathbf{p}_i\right|^3} \right) \cdot d\mathbf{r} = kq \sum_{i=1}^{n} \left(Q_i \int_\gamma \frac{\mathbf{r} - \mathbf{p}_i}{\left|\mathbf{r} - \mathbf{p}_i\right|^3} \cdot d\mathbf{r} \right)$$

Now for each i, direct computation shows that

$$\frac{\mathbf{r} - \mathbf{p}_i}{\left|\mathbf{r} - \mathbf{p}_i\right|^3} = -\nabla \frac{1}{\left|\mathbf{r} - \mathbf{p}_i\right|}.$$

Thus, continuing from above and using the gradient theorem,

$$W = -kq \sum_{i=1}^{n} \left(Q_i \int_\gamma \nabla \frac{1}{\left|\mathbf{r} - \mathbf{p}_i\right|} \cdot d\mathbf{r} \right) = kq \sum_{i=1}^{n} Q_i \left(\frac{1}{\left|\mathbf{a} - \mathbf{p}_i\right|} - \frac{1}{\left|\mathbf{b} - \mathbf{p}_i\right|} \right)$$

We are finished. Of course, we could have easily completed this calculation using the powerful language of electrostatic potential or electrostatic potential energy (with the familiar formulas $W = -\Delta U = -q\Delta V$). However, we have not yet defined potential or potential energy, because the converse of the gradient theorem is required to prove that these are well-defined, differentiable functions and that these formulas hold (see below). Thus, we have solved this problem using only Coulomb's Law, the definition of work, and the gradient theorem

Divergence theorem

"Gauss's theorem" redirects here. For Gauss's theorem concerning the electric field, see Gauss's law.

"Ostrogradsky theorem" redirects here. For Ostrogradsky's theorem concerning the linear instability of the Hamiltonian associated with a Lagrangian dependent on higher time derivatives than the first, see Ostrogradsky instability.

In tensor calculus, the divergence theorem, also known as Gauss's theorem or Ostrogradsky's theorem, is a result that relates the flow (that is, flux) of a tensor field through a surface to the behaviour of the tensor field inside the surface.

More precisely, the divergence theorem states that the outward flux of a tensor field through a closed surface is equal to the volume integral of the divergence over the region inside the surface. Intuitively, it states that the sum of all sources (with sinks regarded as negative sources) gives the net flux out of a region.

The divergence theorem is an important result for the mathematics of physics and engineering, in particular in electrostatics and fluid dynamics.

In physics and engineering, the divergence theorem is usually applied in three dimensions. However, it generalizes to any number of dimensions. In one dimension, it is equivalent to the fundamental theorem of calculus. In two dimensions, it is equivalent to Green's theorem.

The theorem is a special case of the more general Stokes' theorem.

Intuition :

If a fluid is flowing in some area, then the rate at which fluid flows out of a certain region within that area can be calculated by adding up the sources inside the region and subtracting the sinks. The fluid flow is represented by a first order (or a vector) field, and the vector field's divergence at a given point describes the strength of the source or sink there. So, integrating the field's divergence over the interior of the region should equal the integral of the vector field over the region's boundary. The divergence theorem says that this is true.

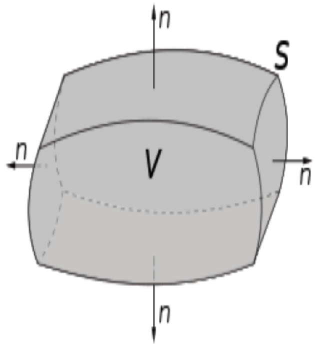
The divergence theorem is employed in any conservation law which states that the volume total of all sinks and sources, that is the volume integral of the divergence, is equal to the net flow across the volume's boundary.

Mathematical statement

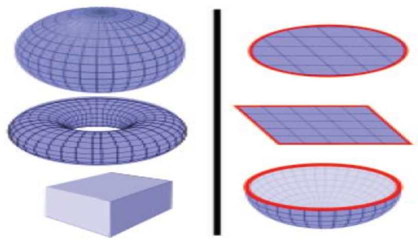
Suppose V is a subset of (in the case of n = 3, V represents a volume in three-dimensional space) which is compact and has a piecewise smooth boundary S (also indicated with ∂V = S). If F is a continuously differentiable vector field defined on a neighbourhood of V, then we have:[6]

$$\iiint_V (\nabla \cdot \mathbf{F})\, dV = \oiint_S (\mathbf{F} \cdot \mathbf{n})\, dS.$$

The left side is a volume integral over the volume V, the right side is the surface integral over the boundary of the volume V. The closed manifold ∂V is quite generally the boundary of V oriented by outward-pointing normals, and n is the outward pointing unit normal field of the boundary ∂V. (dS may be used as a shorthand for ndS.) The symbol within the two integrals stresses once more that ∂V is a closed surface. In terms of the intuitive description above, the left-hand side of the equation represents the total of the sources in the volume V, and the right-hand side represents the total flow across the boundary S.



A region V bounded by the surface $S = \partial V$ with the surface normal n



The divergence theorem can be used to calculate a flux through a closed surface that fully encloses a volume, like any of the surfaces on the left. It can *not* directly be used to calculate the flux through surfaces with boundaries, like those on the right. (Surfaces are blue, boundaries are red.)

<u>Corollaries</u> :

By replacing in the divergence theorem with specific forms, other useful identities can be derived (cf. <u>vector identities</u>).

- *With $F \to F_g$ for a scalar function g and a vector field F,*

$$\iiint_V \left[\mathbf{F} \cdot (\nabla g) + g\,(\nabla \cdot \mathbf{F})\right] dV = \oiint_S g\mathbf{F} \cdot \mathbf{n}\, dS.$$

A special case of this is F = ∇f, in which case the theorem is the basis for Green's identities.

- *With F→F×G for two vector fields F and G,*

$$\iiint_V [\mathbf{G}\cdot(\nabla\times\mathbf{F}) - \mathbf{F}\cdot(\nabla\times\mathbf{G})]\, dV = \oint\!\!\!\oint_S (\mathbf{F}\times\mathbf{G})\cdot \mathbf{n} d\mathbf{S}.$$

- *With F→f_c for a scalar function f and vector field c*

$$\iiint_V \mathbf{c}\cdot\nabla f\, dV = \oint\!\!\!\oint_S (\mathbf{c}f)\cdot \mathbf{n}d\mathbf{S} - \iiint_V f(\nabla\cdot\mathbf{c})\, dV.$$

The last term on the right vanishes for constant C or any divergence free (solenoidal) vector field, e.g. Incompressible flows without sources or sinks such as phase change or chemical reactions etc. In particular, taking C to be constant:

$$\iiint_V \nabla f\, dV = \oint\!\!\!\oint_S f d\mathbf{S}.$$

- *With F→C×F for vector field F and constant vector c*

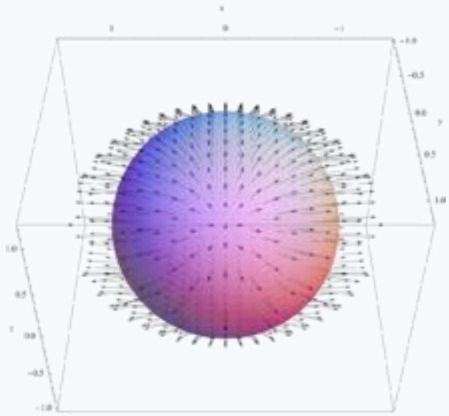
By reordering the triple product on the right hand side and taking out the constant vector of the integral,

$$\iiint_V (\nabla \times \mathbf{F})\, dV \cdot \mathbf{c} = \oiint_S (d\mathbf{S} \times \mathbf{F}) \cdot \mathbf{c}.$$

Hence,

$$\iiint_V (\nabla \times \mathbf{F})\, dV = \oiint_S \mathbf{n} \times \mathbf{F} dS.$$

Example



The vector field corresponding to the example shown. Note, vectors may point into or out of the sphere.

Suppose we wish to evaluate

$$\oiint_S \mathbf{F} \cdot \mathbf{n}\, dS,$$

where S is the unit sphere defined by

$$S=\left\{(x,y,z)\in\mathbb{R}^3\ :\ x^2+y^2+z^2=1\right\},$$

and F is the vector field

$$\mathbf{F}=2x\mathbf{i}+y^2\mathbf{j}+z^2\mathbf{k}.$$

The direct computation of this integral is quite difficult, but we can simplify the derivation of the result using the divergence theorem, because the divergence theorem says that the integral is equal to:

$$\iiint_W(\nabla\cdot\mathbf{F})\,dV=2\iiint_W(1+y+z)\,dV=2\iiint_W dV+2\iiint_W y\,dV+2\iiint_W z\,dV,$$

where W is the unit ball:

$$W=\left\{(x,y,z)\in\mathbb{R}^3\ :\ x^2+y^2+z^2\le 1\right\}.$$

Since the function y is positive in one hemisphere of W and negative in the other, in an equal and opposite way, its total integral over W is zero. The same is true for z:

$$\iiint_W y\,dV=\iiint_W z\,dV=0.$$

Therefore,

$$\oiint_S \mathbf{F}\cdot\mathbf{n}\,dS=2\iiint_W dV=\frac{8\pi}{3},$$

because the unit ball W has volume 4π/3.

Applications :

Differential form and integral form of physical laws

As a result of the divergence theorem, a host of physical laws can be written in both a differential form (where one quantity is the divergence of another) and an integral form (where the flux of one quantity through a closed surface is equal to another quantity). Three examples are Gauss's law (in electrostatics), Gauss's law for magnetism, and Gauss's law for gravity.

Continuity equations

Continuity equations offer more examples of laws with both differential and integral forms, related to each other by the divergence theorem. In fluid dynamics, electromagnetism, quantum mechanics, relativity theory, and a number of other fields, there are continuity equations that describe the conservation of mass, momentum, energy, probability, or other quantities. Generically, these equations state that the divergence of the flow of the conserved quantity is equal to the distribution of sources or sinks of that quantity. The divergence theorem states that any such continuity equation can be written in a differential form (in terms of a divergence) and an integral form (in terms of a flux).

History

The theorem was first discovered by Lagrange in 1762, then later independently rediscovered by Gauss in 1813, by Ostrogradsky, who also gave the first proof of the general theorem, in 1826, by Green in 1828, etc. Subsequently, variations on the divergence theorem are correctly called Ostrogradsky's theorem, but also commonly Gauss's theorem, or Green's theorem.

Examples

To verify the planar variant of the divergence theorem for a region R:

$$R = \left\{(x, y) \in \mathbb{R}^2 \ : \ x^2 + y^2 \leq 1\right\},$$

and the vector field:

$$\mathbf{F}(x, y) = 2y\mathbf{i} + 5x\mathbf{j}.$$

The boundary of R is the unit circle, C, that can be represented parametrically by:

$$x = \cos(s), \quad y = \sin(s)$$

such that $0 \leq s \leq 2\pi$ where s units is the length arc from the point $s = 0$ to the point P on C. Then a vector equation of C is

$$C(s) = \cos(s)\mathbf{i} + \sin(s)\mathbf{j}.$$

At a point P on C:

$$P = (\cos(s), \sin(s)) \ \Rightarrow \ \mathbf{F} = 2\sin(s)\mathbf{i} + 5\cos(s)\mathbf{j}.$$

Therefore,

$$\oint_C \mathbf{F}\cdot\mathbf{n}\,ds = \int_0^{2\pi} (2\sin(s)\mathbf{i} + 5\cos(s)\mathbf{j})\cdot(\cos(s)\mathbf{i} + \sin(s)\mathbf{j})\,ds$$
$$= \int_0^{2\pi} (2\sin(s)\cos(s) + 5\sin(s)\cos(s))\,ds$$
$$= 7\int_0^{2\pi} \sin(s)\cos(s)\,ds$$
$$= 0.$$

Because $M = 2y$, $\frac{\partial M}{\partial x} = 0$, and because $N = 5x$, $\frac{\partial N}{\partial y} = 0$. Thus

$$\iint_R \nabla\cdot\mathbf{F}\,dA = \iint_R \left(\frac{\partial M}{\partial x} + \frac{\partial N}{\partial y}\right)dA = 0.$$

Because M = 2y, ∂M/∂x = 0, and because N = 5x, ∂N/∂y = 0. Thus

Generalizations

One can use the general Stokes' Theorem to equate the n-dimensional volume integral of the divergence of a vector field F over a region U to the (n − 1)-dimensional surface integral of F over the boundary of U:

$$\underbrace{\int\cdots\int_U}_{n} \nabla\cdot\mathbf{F}\,dV = \underbrace{\oint\cdots\oint_{\partial U}}_{n-1} \mathbf{F}\cdot\mathbf{n}\,dS$$

This equation is also known as the Divergence theorem.

When n = 2, this is equivalent to Green's theorem.

When n = 1, it reduces to the Fundamental theorem of calculus.

Tensor fields

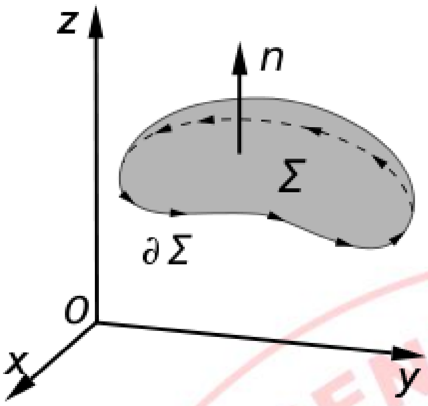
Writing the theorem in Einstein notation:

$$\iiint_V \frac{\partial \mathbf{F}_i}{\partial x_i} dV = \oiint_S \mathbf{F}_i n_i \, dS$$

suggestively, replacing the vector field F with a rank-n tensor field T, this can be generalized to:

$$\iiint_V \frac{\partial T_{i_1 i_2 \cdots i_q \cdots i_n}}{\partial x_{i_q}} dV = \oiint_S T_{i_1 i_2 \cdots i_q \cdots i_n} n_{i_q} \, dS.$$

where on each side, tensor contraction occurs for at least one index. This form of the theorem is still in 3d, each index takes values 1, 2, and 3. It can be generalized further still to higher (or lower) dimensions (for example to 4d spacetime in general relativity[15]).



KELVIN-STOKES THEOREM :

An illustration of the Kelvin–Stokes theorem, with surface Σ, its boundary ∂Σ and the "normal" vector n.

The Kelvin–Stokes theorem (named for Lord Kelvin and George Stokes), also known as the curl theorem, is a theorem in vector calculus on R^3. Given a vector field, the theorem relates the integral of the curl of the vector field over some surface, to the line integral of the vector field around the boundary of the surface. The Kelvin–Stokes theorem is a special case of the "generalized Stokes' theorem. In particular, a vector field on R^3 can be considered as a 1-form in which case curl is the exterior derivative.

If (P(x,y,z),Q(x,y,z),R(x,y,z)) is defined in a region with smooth surface Σ and has first order continuous partial derivatives then

$$\iint_{\Sigma} \left(\left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z} \right) dy\, dz + \left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x} \right) dz\, dx + \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dx\, dy \right)$$

$$= \oint_{\partial\Sigma} \left(P\, dx + Q\, dy + R\, dz \right),$$

where ∂Σ is boundary of region with smooth surface Σ.

Theorem

Let γ: [a, b] → R^2 be a piecewise smooth Jordan plane curve. The Jordan curve theorem implies that γ divides R^2 into two components, a compact one and another that is non-compact. Let D denote the compact part that is bounded by γ and suppose ψ: D → R^3 is smooth, with S := ψ(D). If Γ is the space curvedefined by Γ(t) = ψ(γ(t)) and F is a smooth vector field on R^3, then

Proof

The proof of the theorem consists of 4 steps.[We assume Green's theorem, so what is of concern is how to boil down the three-dimensional complicated problem (Kelvin–Stokes theorem) to a two-dimensional rudimentary problem (Green's theorem). When proving this theorem, mathematicians normally use the differential form. The "pull-back[of a differential form" is a very powerful tool for this situation, but learning differential forms requires substantial background knowledge. So, the proof below does not require knowledge of differential forms, and may be helpful for understanding the notion of differential forms.

First step of the proof (defining the pullback)

Define

$$\mathbf{P}(u,v) = (P_1(u,v), P_2(u,v))$$

so that P is the pull-back of F, and that P(u, v) is R^2-valued function, dependent on two parameters u, v. In order to do so we define P_1 and P_2 as follows.

$$P_1(u,v) = \left\langle \mathbf{F}(\psi(u,v)) \middle| \frac{\partial\psi}{\partial u} \right\rangle, \qquad P_2(u,v) = \left\langle \mathbf{F}(\psi(u,v)) \middle| \frac{\partial\psi}{\partial v} \right\rangle$$

Where <I> is the normal inner product (for Euclidean vectors, the dot product; see Bra-ket notation) of R^3 and *hereinafter, <IAI> stands for the bilinear formaccording to matrix A.*

Second step of the proof (first equation)

According to the definition of a line integral,

$$\begin{aligned}\oint_{\Gamma} \mathbf{F}\cdot d\mathbf{\Gamma} &= \int_a^b \left\langle (\mathbf{F}\circ\Gamma(t)) \middle| \frac{d\Gamma}{dt}(t) \right\rangle dt \\ &= \int_a^b \left\langle (\mathbf{F}\circ\Gamma(t)) \middle| \frac{d(\psi\circ\gamma)}{dt}(t) \right\rangle dt \\ &= \int_a^b \left\langle (\mathbf{F}\circ\Gamma(t)) \middle| (J\psi)_{\gamma(t)} \cdot \frac{d\gamma}{dt}(t) \right\rangle dt\end{aligned}$$

where, Jψ stands for the Jacobian matrix of ψ, and the clear circle denotes function composition. Hence,

$$\begin{aligned}\left\langle (\mathbf{F}\circ\Gamma(t)) \middle| (J\psi)_{\gamma(t)} \frac{d\gamma}{dt}(t) \right\rangle &= \left\langle (\mathbf{F}\circ\Gamma(t)) \middle| (J\psi)_{\gamma(t)} \middle| \frac{d\gamma}{dt}(t) \right\rangle \\ &= \left\langle ({}^t\mathbf{F}\circ\Gamma(t)) \cdot (J\psi)_{\gamma(t)} \middle| \frac{d\gamma}{dt}(t) \right\rangle \\ &= \left\langle \left(\left\langle (\mathbf{F}(\psi(\gamma(t)))) \middle| \frac{\partial\psi}{\partial u}(\gamma(t)) \right\rangle, \left\langle (\mathbf{F}(\psi(\gamma(t)))) \middle| \frac{\partial\psi}{\partial v}(\gamma(t)) \right\rangle \right) \middle| \frac{d\gamma}{dt}(t) \right\rangle \\ &= \left\langle (P_1(u,v), P_2(u,v)) \middle| \frac{d\gamma}{dt}(t) \right\rangle \\ &= \left\langle \mathbf{P}(u,v) \middle| \frac{d\gamma}{dt}(t) \right\rangle\end{aligned}$$

So, we obtain the following equation

$$\oint_{\Gamma} \mathbf{F}\cdot d\mathbf{\Gamma} = \oint_{\gamma} \mathbf{P}\, d\gamma$$

Third step of the proof (second equation)

First, calculate the partial derivatives, using the Leibniz rule (product rule):

$$\frac{\partial P_1}{\partial v} = \left\langle \frac{\partial(\mathbf{F}\circ\psi)}{\partial v} \middle| \frac{\partial \psi}{\partial u} \right\rangle + \left\langle \mathbf{F}\circ\psi \middle| \frac{\partial^2 \psi}{\partial v \partial u} \right\rangle$$

$$\frac{\partial P_2}{\partial u} = \left\langle \frac{\partial(\mathbf{F}\circ\psi)}{\partial u} \middle| \frac{\partial \psi}{\partial v} \right\rangle + \left\langle \mathbf{F}\circ\psi \middle| \frac{\partial^2 \psi}{\partial u \partial v} \right\rangle$$

So,

$$\begin{aligned}
\frac{\partial P_1}{\partial v} - \frac{\partial P_2}{\partial u} &= \left\langle \frac{\partial(\mathbf{F}\circ\psi)}{\partial v} \middle| \frac{\partial \psi}{\partial u} \right\rangle - \left\langle \frac{\partial(\mathbf{F}\circ\psi)}{\partial u} \middle| \frac{\partial \psi}{\partial v} \right\rangle \\
&= \left\langle (J\mathbf{F})_{\psi(u,v)} \cdot \frac{\partial \psi}{\partial v} \middle| \frac{\partial \psi}{\partial u} \right\rangle - \left\langle (J\mathbf{F})_{\psi(u,v)} \cdot \frac{\partial \psi}{\partial u} \middle| \frac{\partial \psi}{\partial v} \right\rangle && \text{chain rule} \\
&= \left\langle \frac{\partial \psi}{\partial u} \middle| (J\mathbf{F})_{\psi(u,v)} \middle| \frac{\partial \psi}{\partial v} \right\rangle - \left\langle \frac{\partial \psi}{\partial u} \middle| {}^t(J\mathbf{F})_{\psi(u,v)} \middle| \frac{\partial \psi}{\partial v} \right\rangle \\
&= \left\langle \frac{\partial \psi}{\partial u} \middle| (J\mathbf{F})_{\psi(u,v)} - {}^t(J\mathbf{F})_{\psi(u,v)} \middle| \frac{\partial \psi}{\partial v} \right\rangle \\
&= \left\langle \frac{\partial \psi}{\partial u} \middle| \left((J\mathbf{F})_{\psi(u,v)} - {}^t(J\mathbf{F})_{\psi(u,v)}\right) \cdot \frac{\partial \psi}{\partial v} \right\rangle \\
&= \left\langle \frac{\partial \psi}{\partial u} \middle| (\nabla \times \mathbf{F}) \times \frac{\partial \psi}{\partial v} \right\rangle && \left((J\mathbf{F})_{\psi(u,v)} - {}^t(J\mathbf{F})_{\psi(u,v)}\right) \cdot \mathbf{x} = (\nabla \times \mathbf{F}) \times \mathbf{x} \\
&= \det\left[(\nabla \times \mathbf{F})(\psi(u,v)) \quad \frac{\partial \psi}{\partial u}(u,v) \quad \frac{\partial \psi}{\partial v}(u,v)\right] && \text{scalar triple product}
\end{aligned}$$

On the other hand, according to the definition of a surface integral,

$$\begin{aligned}
\iint_S (\nabla \times \mathbf{F})\, dS &= \iint_D \left\langle (\nabla \times \mathbf{F})(\psi(u,v)) \middle| \frac{\partial \psi}{\partial u}(u,v) \times \frac{\partial \psi}{\partial v}(u,v) \right\rangle du\, dv \\
&= \iint_D \det\left[(\nabla \times \mathbf{F})(\psi(u,v)) \quad \frac{\partial \psi}{\partial u}(u,v) \quad \frac{\partial \psi}{\partial v}(u,v)\right] du\, dv && \text{scalar triple product}
\end{aligned}$$

So, we obtain

$$\iint_S (\nabla \times \mathbf{F})\, dS = \iint_D \left(\frac{\partial P_2}{\partial u} - \frac{\partial P_1}{\partial v} \right) du\, dv$$

Fourth step of the proof (reduction to Green's theorem)

et C be a positively oriented, piecewise smooth, simple closed curve in a plane, and let D be the region bounded by C.

If L and M are functions of (x, y) defined on an open region containing D and have continuous partial derivatives there, then

$$\oint_C (L\,dx + M\,dy) = \iint_D \left(\frac{\partial M}{\partial x} - \frac{\partial L}{\partial y} \right) dx\,dy$$

where the path of integration along C is anticlockwise.

$$\oint_C L\,dx = \iint_D \left(-\frac{\partial L}{\partial y} \right) dA \qquad (1)$$

and

$$\oint_C M\,dy = \iint_D \left(\frac{\partial M}{\partial x} \right) dA \qquad (2)$$

Assume region D is a type I region and can thus be characterized, as pictured on the right, by

$$D = \{(x, y) \mid a \le x \le b, g_1(x) \le y \le g_2(x)\}$$

where g_1 and g_2 are <u>*continuous functions*</u> *on [a, b].*
Compute the double integral in (1):

$$\begin{aligned} \iint_D \frac{\partial L}{\partial y}\,dA &= \int_a^b \int_{g_1(x)}^{g_2(x)} \frac{\partial L}{\partial y}(x, y)\,dy\,dx \\ &= \int_a^b \Big\{ L(x, g_2(x)) - L(x, g_1(x)) \Big\}\,dx. \qquad (3) \end{aligned}$$

Now compute the line integral in (1). C can be rewritten as the union of four curves: C_1, C_2, C_3, C_4.

With C_1, use the parametric equations: $x = x$, $y = g_1(x)$, $a \le x \le b$. Then

$$\int_{C_1} L(x, y)\, dx = \int_a^b L(x, g_1(x))\, dx.$$

With C_3, use the parametric equations: $x = x$, $y = g_2(x)$, $a \le x \le b$. Then

$$\int_{C_3} L(x, y)\, dx = -\int_{-C_3} L(x, y)\, dx = -\int_a^b L(x, g_2(x))\, dx.$$

The integral over C_3 is negated because it goes in the negative direction from b to a, as C is oriented positively (anticlockwise). On C_2 and C_4, x remains constant, meaning

$$\int_{C_4} L(x, y)\, dx = \int_{C_2} L(x, y)\, dx = 0.$$

Therefore,

$$\begin{aligned}\int_C L\, dx &= \int_{C_1} L(x,y)\, dx + \int_{C_2} L(x,y)\, dx + \int_{C_3} L(x,y)\, dx + \int_{C_4} L(x,y)\, dx \\ &= \int_a^b L(x, g_1(x))\, dx - \int_a^b L(x, g_2(x))\, dx. \qquad (4)\end{aligned}$$

Combining (3) with (4), we get (1) for regions of type I. A similar treatment yields (2) for regions of *type II. Putting the two together, we get the result for regions of type III.*

$$\oint_C (L\,dx + M\,dy) = \iint_D \left(\frac{\partial M}{\partial x} - \frac{\partial L}{\partial y}\right) dx\,dy$$

Green's theorem :

Theorem

Let C be a positively oriented, piecewise smooth, simple closed curve in a plane, and let D be the region bounded by C. If L and M are functions of (x, y) defined on an open region containing D and have continuous partial derivatives there, then

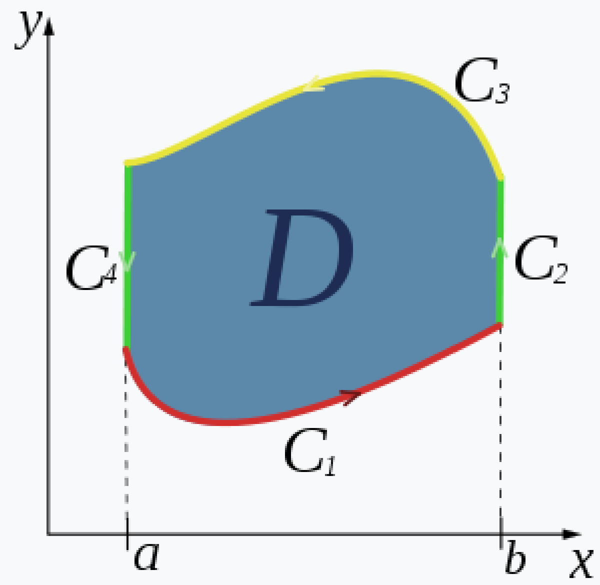
$$\oint_C (L\,dx + M\,dy) = \iint_D \left(\frac{\partial M}{\partial x} - \frac{\partial L}{\partial y}\right) dx\,dy$$

where the path of integration along C is anticlockwise.

In physics, Green's theorem finds many applications. One of which is solving two-dimensional flow integrals, stating that the sum of fluid outflows from a volume is equal to the total outflow summed about an enclosing area. In plane geometry, and in particular, area surveying, Green's theorem can be used to determine the area and centroid of plane figures solely by integrating over the perimeter.

Proof

when D is a simple region



If D is a simple type I region with its boundary consisting of the curves C_1, C_2, C_3, C_4, half of Green's theorem can be demonstrated.

The following is a proof of half of the theorem for the simplified area D, a type I region where C_1 and C_3 are curves connected by vertical lines (possibly of zero length). A similar proof exists for the other half of the theorem when Dis a type II region where C_2 and C_4 are curves connected by horizontal lines (again, possibly of zero length). Putting these two parts together, the theorem is thus proven for regions of type III (defined as regions which are both type I and type II). The general case can then be deduced from this special case by decomposing D into a set of type III regions.

If it can be shown that if

$$\oint_C L\,dx = \iint_D \left(-\frac{\partial L}{\partial y}\right) dA \qquad (1)$$

and

$$\oint_C M\,dy = \iint_D \left(\frac{\partial M}{\partial x}\right) dA \qquad (2)$$

are true, then Green's theorem follows immediately for the region D. We can prove (1) easily for regions of type I, and (2) for regions of type II. Green's theorem then follows for regions of type III.

Assume region D is a type I region and can thus be characterized, as pictured on the right, by

$$D = \{(x, y) \mid a \le x \le b, g_1(x) \le y \le g_2(x)\}$$

where g_1 and g_2 are continuous functions on [a, b]. Compute the double integral in (1):

$$\begin{aligned}\iint_D \frac{\partial L}{\partial y}\,dA &= \int_a^b \int_{g_1(x)}^{g_2(x)} \frac{\partial L}{\partial y}(x, y)\,dy\,dx \\ &= \int_a^b \Big\{L(x, g_2(x)) - L(x, g_1(x))\Big\}\,dx. \qquad (3)\end{aligned}$$

Now compute the line integral in (1). C can be rewritten as the union of four curves: C_1, C_2, C_3, C_4.

With C_1, use the parametric equations: $x = x$, $y = g_1(x)$, $a \le x \le b$. Then

$$\int_{C_1} L(x,y)\,dx = \int_a^b L(x, g_1(x))\,dx.$$

With C_3, use the parametric equations: $x = x$, $y = g_2(x)$, $a \le x \le b$. Then

$$\int_{C_3} L(x,y)\,dx = -\int_{-C_3} L(x,y)\,dx = -\int_a^b L(x, g_2(x))\,dx.$$

The integral over C_3 is negated because it goes in the negative direction from b to a, as C is oriented positively (anticlockwise). On C_2 and C_4, x remains constant, meaning

$$\int_{C_4} L(x,y)\,dx = \int_{C_2} L(x,y)\,dx = 0.$$

Therefore,

$$\begin{aligned}\int_C L\,dx &= \int_{C_1} L(x,y)\,dx + \int_{C_2} L(x,y)\,dx + \int_{C_3} L(x,y)\,dx + \int_{C_4} L(x,y)\,dx \\ &= \int_a^b L(x,g_1(x))\,dx - \int_a^b L(x,g_2(x))\,dx. \qquad (4)\end{aligned}$$

Combining (3) with (4), we get (1) for regions of type I. A similar treatment yields (2) for regions of type II. Putting the two together, we get the result for regions of type III.

result for regions of type III.

$$\oint_C (L\,dx + M\,dy) = \iint_D \left(\frac{\partial M}{\partial x} - \frac{\partial L}{\partial y}\right) dx\,dy$$

APPLICATIONS :

In mathematics

Square Matrices

If the number of columns N of matrix A is equal to its number of rows, then A is a square matrix of order N. The elements aii of a square matrix A form its main diagonal which stretches from top left to bottom right. The diagonal composed of elements aij for which i þ j ¼ N þ 1 is called the cross diagonal and it extends from the bottom left to top right. Square matrices possess properties that are not applicable to other types of matrices such as symmetry and ant symmetry. In addition, many operations such as taking determinants and calculating eigenvalues are only defined for square matrices. The result of multiplying a square matrix of order N by itself is a square matrix of order N. Therefore a square matrix can be multiplied by itself as many times as needed and the notation A^k designates A multiplied by itself k times,

i.e.,

$$\mathbf{A}^k = \underbrace{\mathbf{A} \times \mathbf{A} \times \mathbf{A} \ldots \times \mathbf{A}}_{k\ times}$$

A square matrix A is symmetric if aij = aji i.e., $A^T = A$, and ant symmetric if aij = -aji. An example of a symmetric square matrix of order 3 is

$$\begin{bmatrix} 5 & 3 & -2 \\ 3 & 2 & 7 \\ -2 & 7 & -1 \end{bmatrix}$$

and of an anti symmetric square matrix of order 4 is

$$\begin{bmatrix} 0 & 3 & -2 & 4 \\ -3 & 0 & 1 & -3 \\ 2 & -1 & 0 & -2 \\ -4 & 3 & 2 & 0 \end{bmatrix}$$

A diagonal square matrix D is one for which all elements off the main diagonal are zero while elements on the main diagonal are arbitrary. An example of a square diagonal matrix of order 3 is

A diagonal matrix of order N for which all elements on the main diagonal are 1 (i.e., aii = 1) is called an identity matrix of order N and is designated by I. An identity matrix of order 4 is given by

$$\mathbf{I} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

The inverse of a square matrix A of order N is the square matrix A^{-1} of order N satisfying A

$$\mathbf{A}^{-1}\mathbf{A} = \mathbf{A}\mathbf{A}^{-1} = \mathbf{I}$$

An upper triangular matrix U is a square matrix in which all elements below the main diagonal are zero. Mathematically this can be expressed as

$$\mathbf{U} = \begin{cases} u_{ij} & i \leq j \\ 0 & i > j \end{cases}$$

A lower triangular matrix L is a square matrix in which all elements above the main diagonal are zero. Using mathematical notation, this is written as

$$\mathbf{L} = \begin{cases} \ell_{ij} & i \geq j \\ 0 & i < j \end{cases}$$

Examples of upper and lower triangular square matrices of order 3 as

$$\mathbf{U} = \begin{bmatrix} 1 & 2 & 6 \\ 0 & 4 & 5 \\ 0 & 0 & -7 \end{bmatrix} \quad \mathbf{L} = \begin{bmatrix} 3 & 0 & 0 \\ -1 & 2 & 0 \\ -9 & -2 & 4 \end{bmatrix}$$

<u>Using Matrices to Describe Systems of Equations</u>

Matrices can be used to compactly describe systems of equations . A system of N equations in N unknowns can be written as

$$\begin{array}{l} a_{11}\phi_1 + a_{12}\phi_2 + a_{13}\phi_3 + \ldots + a_{1N}\phi_N = b_1 \\ a_{21}\phi_1 + a_{22}\phi_2 + a_{23}\phi_3 + \ldots + a_{2N}\phi_N = b_2 \\ a_{31}\phi_1 + a_{32}\phi_2 + a_{33}\phi_3 + \ldots + a_{3N}\phi_N = b_3 \\ \quad \vdots \qquad \vdots \qquad \vdots \qquad\qquad \vdots \qquad \vdots \\ a_{N1}\phi_1 + a_{N2}\phi_2 + a_{N3}\phi_3 + \ldots + a_{NN}\phi_N = b_N \end{array}$$

in matrix notation, this system of equations is equivalent to

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & \cdots & a_{1N} \\ a_{21} & a_{22} & a_{23} & \cdots & \cdots & a_{2N} \\ a_{31} & a_{32} & a_{33} & \cdots & \cdots & a_{3N} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{N1} & a_{N2} & a_{N3} & \cdots & \cdots & a_{NN} \end{bmatrix} \begin{bmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \\ \vdots \\ \vdots \\ \phi_N \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ \vdots \\ \vdots \\ b_N \end{bmatrix}$$

or in compact form as

$$\mathbf{A}\boldsymbol{\phi} = \mathbf{b}$$

Tensors and Tensor Operations :

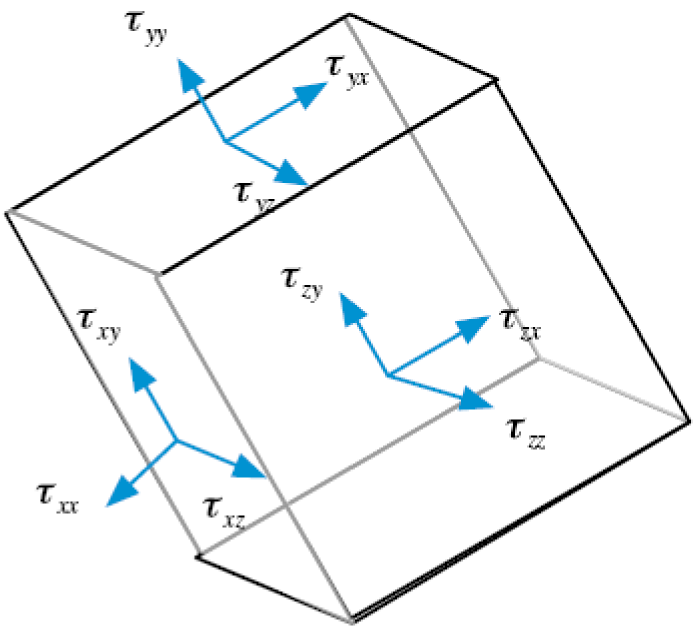
Tensors can be thought of as extensions to the ideas already used when defining quantities like scalars and vectors [2, 20, 21]. A scalar is a tensor of rank zero, and a vector is a tensor of rank one. Tensors of higher rank (2, 3, etc.) can be developed and their main use is to manipulate and transform sets of equations. Since within the scope of this book only tensors of rank two are needed, they will be referred to simply as tensors. Similar to the flow velocity vector v, the deviatoric stress tensor s will be referred to frequently in this book and is used here to illustrate tensor operations. Let x; y; and z represent the directions in an orthonormal Cartesian coordinate system, then the stress tensor s and its transpose designated with superscript $^{T(\tau^T)}$ are represented in terms of their components as

$$\boldsymbol{\tau} = \begin{bmatrix} \tau_{xx} & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \tau_{yy} & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \tau_{zz} \end{bmatrix} \quad \boldsymbol{\tau}^{\mathrm{T}} = \begin{bmatrix} \tau_{xx} & \tau_{yx} & \tau_{zx} \\ \tau_{xy} & \tau_{yy} & \tau_{zy} \\ \tau_{xz} & \tau_{yz} & \tau_{zz} \end{bmatrix}$$

Similar to writing a vector in terms of its components, defining the unit vectors i, j, and k in the x; y; and z direction, respectively, the tensor s given by can be written in terms of its components as

$$\boldsymbol{\tau} = \mathbf{ii}\tau_{xx} + \mathbf{ij}\tau_{xy} + \mathbf{ik}\tau_{xz} + \mathbf{ji}\tau_{yx} + \mathbf{jj}\tau_{yy} + \mathbf{jk}\tau_{yz} + \mathbf{ki}\tau_{zx} + \mathbf{kj}\tau_{zy} + \mathbf{kk}\tau_{zz}$$

allows defining a third type of vector product for multiplying two vectors, known as the dyadic product, and resulting in a tensor with its components formed by ordered pairs of the twovectors. In specific, the dyadic product



of a vector v by itself, arising in the formulation of the momentum equation of fluid flow, gives

$$\left.\begin{aligned}\{\mathbf{vv}\} &= (u\mathbf{i}+v\mathbf{j}+w\mathbf{k})(u\mathbf{i}+v\mathbf{j}+w\mathbf{k})\\ &= \mathbf{ii}uu+\mathbf{ij}uv+\mathbf{ik}uw+\\ &\quad \mathbf{ji}vu+\mathbf{jj}vv+\mathbf{jk}vw+\\ &\quad \mathbf{ki}wu+\mathbf{kj}wv+\mathbf{kk}ww\end{aligned}\right\} \Rightarrow \{\mathbf{vv}\} = \begin{bmatrix} uu & uv & uw \\ vu & vv & vw \\ wu & wv & ww \end{bmatrix}$$

The gradient of a vector v is a tensor given by

$$\left.\begin{aligned}\{\nabla\mathbf{v}\} &= \left(\frac{\partial}{\partial x}\mathbf{i}+\frac{\partial}{\partial y}\mathbf{j}+\frac{\partial}{\partial z}\mathbf{k}\right)(u\mathbf{i}+v\mathbf{j}+w\mathbf{k})\\ &= \mathbf{ii}\frac{\partial u}{\partial x}+\mathbf{ij}\frac{\partial v}{\partial x}+\mathbf{ik}\frac{\partial w}{\partial x}+\\ &\quad \mathbf{ji}\frac{\partial u}{\partial y}+\mathbf{jj}\frac{\partial v}{\partial y}+\mathbf{jk}\frac{\partial w}{\partial y}+\\ &\quad \mathbf{ki}\frac{\partial u}{\partial z}+\mathbf{kj}\frac{\partial v}{\partial z}+\mathbf{kk}\frac{\partial w}{\partial z}\end{aligned}\right\} \Rightarrow \{\nabla\mathbf{v}\} = \begin{bmatrix} \frac{\partial u}{\partial x} & \frac{\partial v}{\partial x} & \frac{\partial w}{\partial x} \\ \frac{\partial u}{\partial y} & \frac{\partial v}{\partial y} & \frac{\partial w}{\partial y} \\ \frac{\partial u}{\partial z} & \frac{\partial v}{\partial z} & \frac{\partial w}{\partial z} \end{bmatrix}$$

The sum of two tensors ∂ and T is a tensor ∑ whose components are the sum of the corresponding components of the two tensors, i.e.,

∑ =∂+T=

$$\begin{bmatrix} \sigma_{xx}+\tau_{xx} & \sigma_{xy}+\tau_{xy} & \sigma_{xz}+\tau_{xz} \\ \sigma_{yx}+\tau_{yx} & \sigma_{yy}+\tau_{yy} & \sigma_{yz}+\tau_{yz} \\ \sigma_{zx}+\tau_{zx} & \sigma_{zy}+\tau_{zy} & \sigma_{zz}+\tau_{zz} \end{bmatrix}$$

Multiplying a tensor s by a scalar s results in a tensor whose components are multiplied by that scalar, i.e.,

$$\{s\boldsymbol{\tau}\} = \begin{bmatrix} s\tau_{xx} & s\tau_{xy} & s\tau_{xz} \\ s\tau_{yx} & s\tau_{yy} & s\tau_{yz} \\ s\tau_{zx} & s\tau_{zy} & s\tau_{zz} \end{bmatrix}$$

The dot product of a tensor s by a vector v results in the following vector:

$$[\boldsymbol{\tau}\cdot\mathbf{v}] = \begin{pmatrix} \mathbf{ii}\tau_{xx} + \mathbf{ij}\tau_{xy} + \mathbf{ik}\tau_{xz} + \mathbf{ji}\tau_{yx} + \\ \mathbf{jj}\tau_{yy} + \mathbf{jk}\tau_{yz} + \mathbf{ki}\tau_{zx} + \mathbf{kj}\tau_{zy} + \mathbf{kk}\tau_{zz} \end{pmatrix} \cdot (u\mathbf{i} + v\mathbf{j} + w\mathbf{k})$$

which upon expanding becomes

$$\begin{aligned} [\boldsymbol{\tau}\cdot\mathbf{v}] = {} & \mathbf{ii}\cdot\mathbf{i}\tau_{xx}u + \mathbf{ii}\cdot\mathbf{j}\tau_{xx}v + \mathbf{ii}\cdot\mathbf{k}\tau_{xx}w + \mathbf{ij}\cdot\mathbf{i}\tau_{xy}u + \mathbf{ij}\cdot\mathbf{j}\tau_{xy}v \\ & + \mathbf{ij}\cdot\mathbf{k}\tau_{xy}w + \mathbf{ik}\cdot\mathbf{i}\tau_{xz}u + \mathbf{ik}\cdot\mathbf{j}\tau_{xz}v + \mathbf{ik}\cdot\mathbf{k}\tau_{xz}w + \mathbf{ji}\cdot\mathbf{i}\tau_{yx}u \\ & + \mathbf{ji}\cdot\mathbf{j}\tau_{yx}v + \mathbf{ji}\cdot\mathbf{k}\tau_{yx}w + \mathbf{jj}\cdot\mathbf{i}\tau_{yy}u + \mathbf{jj}\cdot\mathbf{j}\tau_{yy}v + \mathbf{jj}\cdot\mathbf{k}\tau_{yy}w \\ & + \mathbf{jk}\cdot\mathbf{i}\tau_{yz}u + \mathbf{jk}\cdot\mathbf{j}\tau_{yz}v + \mathbf{jk}\cdot\mathbf{k}\tau_{yz}w + \mathbf{ki}\cdot\mathbf{i}\tau_{zx}u + \mathbf{ki}\cdot\mathbf{j}\tau_{zx}v \\ & + \mathbf{ki}\cdot\mathbf{k}\tau_{zx}w + \mathbf{kj}\cdot\mathbf{i}\tau_{zy}u + \mathbf{kj}\cdot\mathbf{j}\tau_{zy}v + \mathbf{kj}\cdot\mathbf{k}\tau_{zy}w + \mathbf{kk}\cdot\mathbf{i}\tau_{zz}u \\ & + \mathbf{kk}\cdot\mathbf{j}\tau_{zz}v + \mathbf{kk}\cdot\mathbf{k}\tau_{zz}w \end{aligned}$$

Using

$$[\boldsymbol{\tau}\cdot\mathbf{v}] = (\tau_{xx}u + \tau_{xy}v + \tau_{xz}w)\mathbf{i} + (\tau_{yx}u + \tau_{yy}v + \tau_{yz}w)\mathbf{j} + (\tau_{zx}u + \tau_{zy}v + \tau_{zz}w)\mathbf{k}$$

The above equation can be derived using matrix multiplication as

$$[\boldsymbol{\tau} \cdot \mathbf{v}] = \begin{bmatrix} \tau_{xx} & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \tau_{yy} & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \tau_{zz} \end{bmatrix} \begin{bmatrix} u \\ v \\ w \end{bmatrix} = \begin{bmatrix} \tau_{xx}u + \tau_{xy}v + \tau_{xz}w \\ \tau_{yx}u + \tau_{yy}v + \tau_{yz}w \\ \tau_{zx}u + \tau_{zy}v + \tau_{zz}w \end{bmatrix}$$

In a similar way the divergence of a tensor t is found to be a vector given by

$$[\nabla \cdot \boldsymbol{\tau}] = \left(\frac{\partial \tau_{xx}}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + \frac{\partial \tau_{zx}}{\partial z}\right)\mathbf{i} + \left(\frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \tau_{yy}}{\partial y} + \frac{\partial \tau_{zy}}{\partial z}\right)\mathbf{j} + \left(\frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} + \frac{\partial \tau_{zz}}{\partial z}\right)\mathbf{k}$$

The double dot product of two tensors t and {Δv} is a scalar computed as

$$(\boldsymbol{\tau} : \nabla \mathbf{v}) = \begin{pmatrix} \mathbf{ii}\tau_{xx} + \mathbf{ij}\tau_{xy} + \mathbf{ik}\tau_{xz} + \\ \mathbf{ji}\tau_{yx} + \mathbf{jj}\tau_{yy} + \mathbf{jk}\tau_{yz} + \\ \mathbf{ki}\tau_{zx} + \mathbf{kj}\tau_{zy} + \mathbf{kk}\tau_{zz} \end{pmatrix} : \begin{pmatrix} \mathbf{ii}\frac{\partial u}{\partial x} + \mathbf{ij}\frac{\partial v}{\partial x} + \mathbf{ik}\frac{\partial w}{\partial x} + \\ \mathbf{ji}\frac{\partial u}{\partial y} + \mathbf{jj}\frac{\partial v}{\partial y} + \mathbf{jk}\frac{\partial w}{\partial y} + \\ \mathbf{ki}\frac{\partial u}{\partial z} + \mathbf{kj}\frac{\partial v}{\partial z} + \mathbf{kk}\frac{\partial w}{\partial z} \end{pmatrix}$$

The final value is obtained by expanding the above product and performing the double dot product on the various terms. For example,

$$\mathbf{ij}\tau_{xy} : \mathbf{ji}\frac{\partial u}{\partial y} = \mathbf{i}\underbrace{\mathbf{j} : \mathbf{j}}_{=1}\mathbf{i}\tau_{xy}\frac{\partial u}{\partial y} = \underbrace{\mathbf{i} \cdot \mathbf{i}}_{=1}\tau_{xy}\frac{\partial u}{\partial y} = \tau_{xy}\frac{\partial u}{\partial y}$$

Performing the same steps on every term in the expanded product, the final form of (t,Δv) is obtained as

$$(\boldsymbol{\tau} : \nabla \mathbf{v}) = \tau_{xx}\frac{\partial u}{\partial x} + \tau_{xy}\frac{\partial u}{\partial y} + \tau_{xz}\frac{\partial u}{\partial z} + \tau_{yx}\frac{\partial v}{\partial x} + \tau_{yy}\frac{\partial v}{\partial y} + \tau_{yz}\frac{\partial v}{\partial z} + \tau_{zx}\frac{\partial w}{\partial x} + \tau_{zy}\frac{\partial w}{\partial y} + \tau_{zz}\frac{\partial w}{\partial z}$$

<u>Uses of vector integration is on computer science and engineering</u>

Because vectors and matrices are used in linear algebra, anything that requires the use of arrays that are linear dependent requires vectors. A few well-known examples are:

- Internet search
- Graph analysis
- Machine learning
- Graphics
- Bioinformatics
- Scientific computing
- Data mining
- Computer vision
- Speech recognition
- Compilers
- Parallel computing

Vector is one of the most important concepts in Physics. It scope and usage extends to every corner of physics, from the very small i.e. quantum reams to the very fast i.e. relativity, encompassing everything that lies in this broad domain.

- On the Newtonian level, the motion of bodies is understood in terms of position, velocity, momentum vectors for translation motion and associated vectors for other kinds of motion.

- The whole of quantum physics is built on a vector space known as Hilbert space. The state of system is represented by a vector which resides in this space.
- In Special theory of relativity, the motion of body is studied in terms of four-vectors in the space-time basis. General theory of relativity goes beyond vectors into a more generalized mathematical structure known as tensor.

We have seen how integration can be used to find an area between a curve and the xx-axis. With very little change we can find some areas between curves; indeed, the area between a curve and the xx-axis may be interpreted as the area between the curve and a second "curve" with equation y=0y=0. In the simplest of cases, the idea is quite easy to understand.

Whether we are interested in a function as a purely mathematical object or in connection with some application to the real world, it is often useful to know what the graph of the function looks like. We can obtain a good picture of the graph using certain crucial information provided by derivatives of the function and certain limits.

VECTOR INTEGRATION USE IN INTERNET SERCH:

A retina- The most of the text-retrieval techniques are based on indexing keywords. Since only keywords are unable to capturing the whole documents' content, they results poor retrieval performance. But indexing of keywords is still the most applicable way to process large corpora of text. After identification of the significant index term a document can be matched to a given query by Boolean Model or Statistical Model. Boolean Model applies a match that relies on the extent. Fig.1 represents the documents Doc1 and Doc2 in space of three terms namely "Information", "Retrieval" and "System". Three are perpendicular dimensions for each term represents "Term-

Independence”. This independence can be of two types namely linguistic and statistical. When the occurrence of a single term does not depend upon appearance of other term, it is called Statistical independence. In Linguistic independence; interpretation of a term does not rely on other any term an index term satisfies a Boolean expression while statistical properties are used to discover similarity between query and document in Statistical Model. The statistically based “Vector Space Model” which is based on the theme of placing the documents in the dimensional space, where n is number of distinct terms or words (as- t1, t2…tn) which constitutes the whole vocabulary of the corpus or text collection. Each dimension belongs to a particular term. Each document is considered as a vector as- D1, D2…Dr; where r is the total number of documents in corpora. Document Vector can be shown as following: Dr={d1r,d2r,d3r,……..dnr}

PROPOSED SYSTEM:

In the proposed system, we propose a content ontology and location ontology to accommodate the extracted content and location concepts as well as the relationships among the concepts. We introduce different entropies to indicate the amount of concepts associated with a query and how much a user is interested in these concepts. With the entropies, we are able to estimate the effectiveness of personalization for different users and different queries. Based on the proposed ontology’s and entropies. Fig 2- System Architecture design We adopt an SVM to learn personalized ranking functions for content and location preferences. We use the personalization effectiveness to integrate the learned ranking functions into a coherent profile for personalized reran king. We implement a working prototype to validate the proposed ideas. It consists of a middleware for capturing user click through, performing personalization, and interfacing with commercial search engines at the backend. Empirical results show that OMF can successfully capture users' content and location preferences and utilize the preferences to produce relevant results for the users.

Finally, it significantly out-performs strategies which use either content or location preference only. The personalized Meta search engines don't require traversing the network, downloading web documents or building up an index. They are mainly consisted of member search engine selection, query forwarding, result integration and other algorithms. So, compared to robot based search engines or directory based search engines, the personalized Meta search engines have much lower technical doorsill and threshold in development and maintenance. This forces users to manually submit their queries to multiple search engines one after another until they find the information they need or give up their retrieval desire. The architectural design

Vector integration use in Graphics:

Applications used for creating and editing vector graphics are known as *Drawing Packages*. Like bitmap graphics software there are commercial, freeware and open source Vector drawing applications. There are also web-based and free versions of vector drawing applications. Most vector graphics software applications can also be used to create meta graphics.

Adobe Illustrator is the industry leader in vector drawing applications. This was one of the original software applications produced by Adobe. Originally used for creating vector designs for print, it is now also used for creating web and screen based graphics (including graphics for mobile devices). Until 1989 Illustrator was only supported on the MAC operating system. At the time of writing CS3 is the most up-to-date version. This allows for closer integration with Adobe's web authoring and 2D animation application, Adobe Flash.

CorelDraw is another popular vector drawing application. It has many similar features to Illustrator. One of the differences between the two packages is that CorelDraw allows for multi page editing whereas illustrator only allows for single page editing. At the time of writing CorelDraw is part of a suite of applications known as CorelDraw Graphics Suite, which consists of five applications, including a bitmap editor Corel Photo-Paint. The main disadvantage of CorelDraw and possibly one of the reasons that Illustrator is more widely used, is that it is only supported on the Windows operating system (Windows 2000 onwards).

Adobe Freehand (formerly Macromedia) is used for creating vector graphics for desktop publishing and web platforms. It has a lot of similar features to Adobe Illustrator, so much so that Adobe have no plans to upgrade Freehand at present. Instead they are suggesting that customers move over to Illustrator. Freehand is still on sale though at the time of writing.

Adobe Fireworks - see Bitmap graphics software applications section above.

Other vector drawing applications include:

- Adobe Flash - see animation applications
- KAI powertools
- JascWebDraw
- LogoEase- web based software
- The Flame project - Open source software used to create SVG files

Animation Software:

More and more graphics software applications support features for creating simple 2D (dimensional) animated graphics. For example Photoshop can be used to create animated Gifs, eg rollovers. However more complex animations for web and multimedia applications can be produced using specialist software applications. Paint and drawing applications can be used to create the graphics to include in an animation. They would then be imported into one of the following applications and animated.

- **Adobe Flash** is the industry standard application for creating 2D animation for web and multimedia. This can be used to create anything from simple to more complex animation. Bitmaps can be integrated and converted to vectors. Unlike typical graphics applications Flash has the added bonus of being able to produce interactivity using 'Action Script', therefore it is now used for creating web based games.

The original intention of Flash was purely to create 2D web based animation, however, as it has become more popular, its scripting language increased and extra features have been added, it has turned into a 'web authoring tool'. It is now also used to create entire websites. Flash files are vector based. SWF (pronounced *swift*) is the standard published format for files to be incorporated into web pages. Flash player is required to view .swf files.

- **Adobe Director** (formerly Macromedia) is a 'multimedia authoring tool' which is used for creating offline (kiosks, DVDs) and online multimedia applications including 2D animations. Director was devised by designers, developers and animators. As a result it uses many of the same terms and 'tools' that traditional animators work with, eg frames, timelines, key frames, twined frames, scripts. Flash is based largely on Director. The main difference between the two is their scripting languages. Until now, Director's lingo script has been more powerful than Action Script, however the gap is narrowing between the two. More noticeably Director can be used to create 3D animations.

Director used to be more associated with creating disk based multimedia applications and Flash with web content, however, again this is changing. Director produces Shockwave movies for the web. These are used to incorporate highly interactive content within a website, eg games. The Shockwave player is required to view shockwave files.

Other Animation graphics software applications include:

- ToonBoom Animation
- Anime Studio

Vector integration use in physics:

→ **line integral of** ∇ ψ:

- We found that there were various ways of taking derivatives of fields. Some gave vector fields; some gave scalar fields. Although we developed many different formulas, everything in Chapter 2 could be summarized in one rule: the operators ∂/∂x, ∂/∂y, and ∂/∂z are the three components of a vector operator ∇ . We would now like to get some understanding of the significance of the derivatives of fields. We will then have a better feeling for what a vector field equation means.
- We have already discussed the meaning of the gradient operation (∇ on a scalar). Now we turn to the meanings of the divergence and curl operations. The interpretation of these quantities is best done in terms of certain vector integrals and equations relating such integrals. These equations cannot, unfortunately, be obtained from vector algebra by some easy substitution, so you will just have to learn them as something new. Of these integral formulas, one is practically trivial, but the other two are not. We will derive them and explain their implications. The equations we shall study are really mathematical theorems. They will be useful not only for interpreting the meaning and the content of the divergence and the curl, but also in working out general physical theories. These mathematical theorems are, for the theory of fields, what the theorem of the conservation of energy is to the mechanics of particles. General theorems like these are important for a deeper understanding of physics. You will find, though, that they

are not very useful for solving problems—except in the simplest cases. It is delightful, however, that in the beginning of our subject there will be many simple problems which can be solved with the three integral formulas we are going to treat. We will see, however, as the problems get harder, that we can no longer use these simple methods.

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→The circulation of a vector field

We wish now to look at the curl in somewhat the same way we looked at the divergence. We obtained Gauss' theorem by considering the integral over a surface, although it was not obvious at the beginning that we were going to be dealing with the divergence. How did we know that we were supposed to integrate over a surface in order to get the divergence? It was not at all clear that this would be the result. And so with an apparent equal lack of justification, we shall calculate something else about a vector and show that it is related to the curl. This time we calculate what is called the circulation of a vector field. If C is any vector field, we take its component along a curved line and take the integral of this component all the way around a complete loop. The integral is called the *circulation* of the vector field around the loop. We have already considered a line integral of $\nabla \psi$ earlier in this chapter. Now we do the same kind of thing for *any* vector field C.

References

1.Arfken G (1985) Mathematical methods for physicists, 3rd edn. Academic Press, Orlando, FL

2. Aris R (1989) Vectors, tensors, and the basic equations of fluid mechanics. Dover, New York

3. Crowe MJ (1985) A history of vector analysis: the evolution of the idea of a vectorial system. Dover, New York

4. Marsden JE, Tromba AJ (1996) Vector calculus. WH Freeman, New York

5. Jeffreys H, Jeffreys BS (1988) methods of mathematical physics. Cambridge University Press, Cambridge, England

6. Morse PM, Feshbach H (1953) Methods of theoretical physics, Part I. McGraw-Hill, New York

7. Schey HM (1973) Div, grad, curl, and all that: an informal text on vector calculus. Norton, New York

8. Schwartz M, Green S, Rutledge W (1960) A vector analysis with applications to geometry and physics. Harper Brothers, New York

9. M1 Anton H (1987) Elementary linear algebra. Wiley, New York

10. Bretscher O (2005) Linear algebra with applications. Prentice Hall, New Jersey

11. Bronson R (1989), Schaum's outline of theory and problems of matrix operations. McGraw–Hill, New York

12. Arnold VI, Cooke R (1992) Ordinary differential equations. Springer-Verlag, Berlin, DE; New York, NY

13. Horn RA, Johnson CR (1985) Matrix analysis. Cambridge University Press, Cambridge

14. Brown WC (1991) Matrices and vector spaces. Marcel Dekker, New York

15. Golub GH, Van Loan CF (1996) Matrix Computations. Johns Hopkins, Baltimore

16. Greub WH (1975) Linear algebra, graduate texts in mathematics. Springer-Verlag, Berlin, DE; New York, NY

17. Lang S (1987) Linear algebra. Springer-Verlag, Berlin, DE; New York, NY

18. Mirsky L (1990) An introduction to linear algebra. Courier Dover Publications, New York

19. Nering ED (1970) Linear algebra and matrix theory. Wiley, New York

20. Spiegel MR (1959) Schaum's outline of theory and problems of vector analysis and an introduction to tensor analysis. Schaum, New York

21. Heinbockel JH (2001) Introduction to tensor calculus and continuum mechanics. Trafford Publishing, Victoria

22. Williamson R, Trotter H (2004) Multivariable mathematics. Pearson Education, Inc, New York

23. Cauchy A (1846) Sur les intégrales qui s'étendent à tous les points d'unecourbefermée. Comptesrendus

23:251–255 24. Riley KF, Hobson MP, Bence SJ (2010) Mathematical methods for physics and engineering. Cambridge University Press, Cambridge

25. Spiegel MR, Lipschutz S, Spellman D (2009) Vector analysis. Schaum's Outlines, McGraw Hill (USA)

26. Wrede R, Spiegel MR (2010) Advanced calculus. Schaum's Outline Series

27. Katz VJ (1979) The history of stokes's theorem. Math Mag (Math Assoc Am) 52:146–156

28. Morse PM, Feshbach H (1953) Methods of theoretical physics, Part I. McGraw-Hill, New York

29. Stewart J (2008) Vector calculus, Calculus: early transcendentals. Thomson Brooks/Cole, Connecticut

30. Lerner RG, Trigg GL (1994) Encyclopaedia of physics. VHC

31. Byron F, Fuller R (1992) Mathematics of classical and quantum physics. Dover Publications, New York

32. Spiegel MR, Lipschutz S, Spellman D (2009) Vector analysis. Schaum’s Outlines, McGraw Hill

33. Flanders H (1973) Differentiation under the integral sign. Am Math Monthly 80(6):615–627 34. Boros G, Moll V (2004) Irresistible integrals: symbolics, analysis and experiments in the evaluation of integrals. Cambridge University Press, Cambridge, England

35. Hijab O (1997) Introduction to calculus and classical analysis. Springer, New York

36. Kaplan W (1992) Advanced calculus. Addison-Wesley, Reading, MA

D-P. N. Dunford and B. J. Pettis, Linear operations on summable functions, Trans AMS 47 (1940), 323-392. D-S. N. Dunford and J. Schwartz, Linear Operators. Part I. General Theory, Wiley Interscience, New York, 1958. E. R. Elliot, Stochastic Calculus and Applications, Springer, New York, 1982. Fi. D. L. Fisk, Quasimartingales, Trans AMS 120 (1965), 369-389. Fo. H. Fölmer, Quasimartingales ά deux indices, CRAS, Paris 288 (1979), 61-63. G-P. B. Gravereaux and J. Pellaumail, Formuled'Itó des processus a valeursdans des espaces de Banach, Ann. Inst. H. Poincar 10 (1974), 399-422. Go. M. Gowurin, Über die Stieltjessche Integration abstrakterFunktionen, Fund. Math. 27 (1936), 255-268. Gr. M. Green, Planar Stochastic Integration Relative to Quasimartingales, Real and Stochastic Analysis, CRC, Boca Raton, FL, (1997), 65-157. H. P. R. Halmos, Measure Theory, Van Nostrand, New York, 1950.IT.A. and C. IonescuTulcea, Topics in the Theory of Lifting, Springer, New York, 1969.Ka. G. Kallianpur, Stochastic Filtering Theory, Springer, New York, 1980. K-M. N. Karoui and S. Méléard, Martingale measures and stochastic calculus, Prob. Theory and Related Fields 84 (1990), 83-101. Kl.l. I. Kluvanek, On the theory of vector measures I, Mat. Fyz.Casopis 11 (1961), 173-192. K1.2. I. Kluvanek, On the theory of vector measures II, Mat. Fyz.Casopis 11 (1961), 76-81. K1.3. I. Kluvanek, Completion of vector measures, Rev. Roumaine Math.

<u>CONCLUSION:</u>

Differential equations plays major role in applications of sciences and engineering. It arises in wide variety of engineering applications for e.g. electromagnetic theory, signal processing, computational fluid dynamics, etc. These equations can be typically solved using either analytical or numerical methods. Since many of the differential equations arising in real life application cannot be solved analytically or we can say that their analytical solution does not exist. For such type of problems certain numerical methods exists in the literature. In this book, our main focus is to present an emerging meshless method based on the concept of neural networks for solving differential equations or boundary value problems of type ODE's as well as PDE's. Here in this book, we have started with the fundamental concept of differential equation, some real life applications where the problem is arising and explanation of some existing numerical methods for their solution. We have also presented some basic concept of neural network that is required for the study and history of neural networks. Different neural network methods based on multilayer perception, radial basis functions, multiquadric functions and finite element etc. are then presented for solving differential equations. It has been pointed out that the employment of neural network architecture adds many attractive features towards the problem compared to the other existing methods in the literature. Preparation of input data, robustness of methods and the high accuracy of the solutions made these methods highly acceptable. The main advantage of the proposed approach is that once the network is trained, it allows evaluation of the solution at any desired number of points instantaneously with spending negligible computing time. Moreover, different hybrid approaches are also available and the work is in progress to use better optimization algorithms. People are also working in the combination of neural networks to other existing methods to propose a new method for construction of a better trail solution for all kind of boundary value problems. Such a collection could not be exhaustive; indeed, we can hope to give only an indication of what is possible.